The dual Artin group and isomorphism problem

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World of Groupcraft

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Coxeter groups and Artin groups

A Coxeter graph Γ encodes data to define these groups. It has vertices $1, \dots, n$, and labels $m_{ij} \in \{2, 3, \dots, \infty\}$. Write $\Pi(a, b; k) \coloneqq aba\cdots$ (total length k).

Definition

$$W_\Gamma = \left\langle s_1, \dots, s_n \;\middle|\; \Pi(s_i, s_j; m_{ij}) = \Pi(s_j, s_i; m_{ij})\ \forall m_{ij} \neq \infty,\ s_i^2 = 1 \right\rangle \qquad \text{(Coxeter group)}$$

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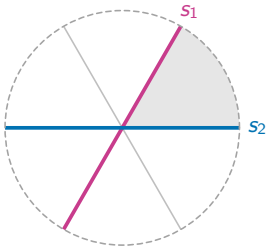
- $A_\Gamma \twoheadrightarrow W_\Gamma$, $t_i \mapsto s_i$.
- XXL type: all $m_{ij} \geq 5$.

Two examples

Reflections $R := \bigcup_w wSw^{-1}$ corresponds the collection of mirrors in each picture.

$\mathcal{S}_3$

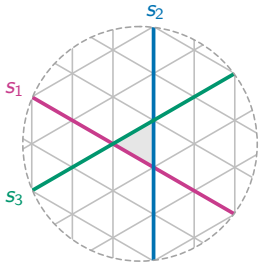




Symmetries of a triangle, $|R| = 3$.

3-3-3





Symmetries of a tiling, $|R| = \infty$.

Intervals and interval groups

Intervals and interval groups

Take a group G, a generating set X and an element $g \in G$.

- The interval $P = [1, g]_X$ is the union of all geodesics from 1 to g in $\mathrm{Cay}(G, X)$.
- The interval group is

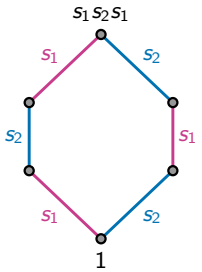
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Example. $G = \mathcal{S}_3$, $X = \{s_1, s_2\}$, $g = s_1 s_2 s_1$:

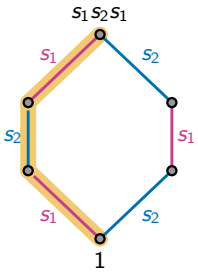
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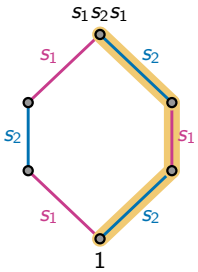
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Garside theory, very roughly

A poset P is a lattice if any two elements have a unique least upper bound and a unique greatest lower bound.



not a lattice

Theorem (Garside '69, Deligne '72, Brieskorn–Saito '72, . . .)

If P is a lattice (plus other conditions), then $G(P)$ is a Garside group: it has solvable word problem, a finite $K(\pi, 1)$, and more.

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- If W_Γ is finite, with longest element w_0, then $G([1, w_0]_S) \cong A_\Gamma$ (e.g. $\mathcal{S}_3$ with $w_0 = s_1 s_2 s_1$).
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- In this case, the Artin group is an interval group of the Coxeter group.
- But w_0 only exists when W_Γ is finite (not for the 3-3-3 group, or XXL Coxeter groups).

The dual structure

This proposes a different Garside structure. We extend the generating set to R, with top element being a Coxeter element $c = s_{\mu(1)} \cdots s_{\mu(n)}$, for $\mu \in \mathcal{S}_n$.

Definition (Birman–Ko–Lee '98, Brady '01, Brady–Watt '02, Bessis '03)

The dual Artin group is $A_\Gamma^\vee(c) := G([1, c]_R)$.

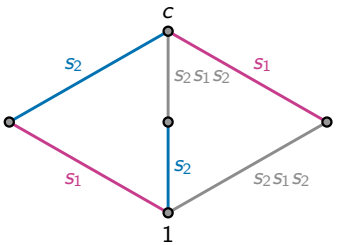
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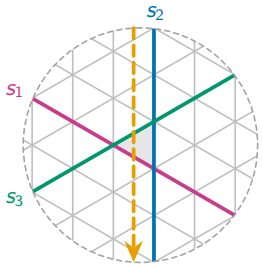
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3-3-3



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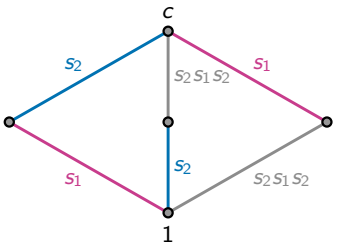
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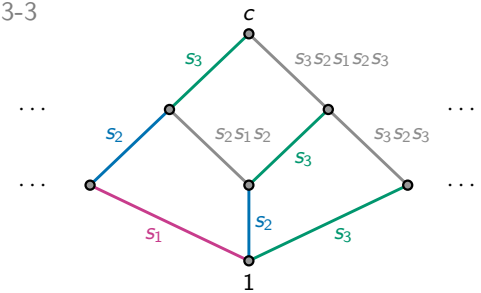
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The isomorphism problem

There is a canonical epimorphism $A_\Gamma \twoheadrightarrow A_\Gamma^\vee(c)$.

Question

Is it an isomorphism, for every Γ and every c?

- Known for W_Γ finite, affine, rank 3, and some free and direct products.

  Brady–Watt '02, Bessis '03; McCammond–Sulway '17, Paolini–Salvetti '21; Delucchi–Paolini–Salvetti '24; Resteghini '24.
- Why care? The dual structure has proved vital in recent $K(\pi, 1)$ proofs (affine, rank 3), even where $[1, c]_R$ is not a lattice.

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Theorem (O '26)

If all $m_{ij} \geq 5$, then $A_\Gamma \cong A_\Gamma^\vee(c)$, for every Coxeter element c.

The Hurwitz action

B_n acts on n-tuples in any group:

$$\sigma_i \cdot (\ldots, g_i, g_{i+1}, \ldots) = (\ldots, g_{i+1}, g_{i+1}^{-1} g_i g_{i+1}, \ldots).$$

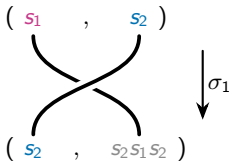
This preserves the product $g_1 \cdots g_n$.

Theorem (Hurwitz 1891, Igusa–Schiffler '10)

B_n acts transitively on the minimal factorisations of c into reflections.

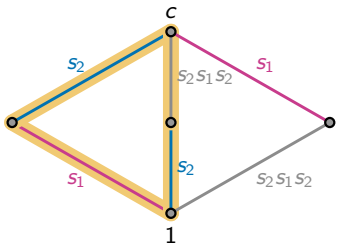
We define HurElt$(g_1, \ldots, g_n)$ to be all entries of all tuples in the orbit.

$\mathcal{S}_3$



Hurwitz moves correspond to relations

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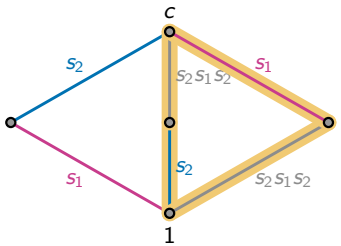


Squares in $[1, c]_R$ are Hurwitz moves, corresponding to the relations like the following.

$$[a][b] = [b][b^{-1}ab]$$
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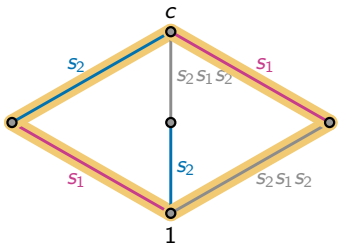


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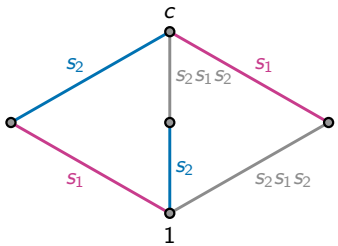


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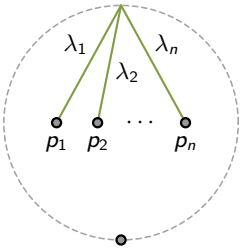
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So for $c = s_1 \cdots s_n$, the group $A_\Gamma^\vee(c)$ is generated by $[s_1], \ldots, [s_n]$, and all of its relations come from the Hurwitz orbit of $(s_1, \ldots, s_n)$.

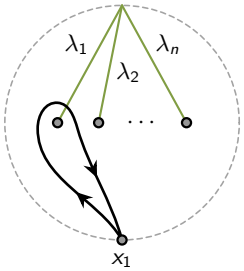
The braid group acting on the free group

- $F_n = \pi_1(n\text{-punctured disc}) = \langle x_1, \ldots, x_n \rangle$.
- Read off a word from the crossings with the green lines: crossing λ_j left to right gives x_j.
- B_n acts on F_n by homeomorphisms of the disc (the Artin action $\star$), e.g. $\sigma_1 \star x_1 = x_1 x_2 x_1^{-1}$.



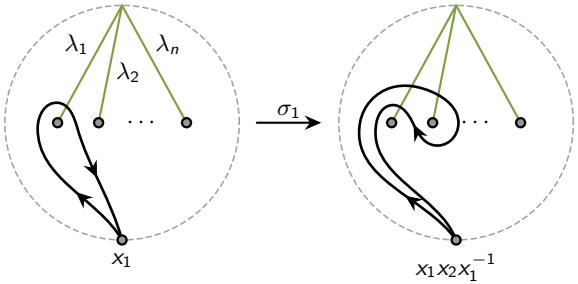
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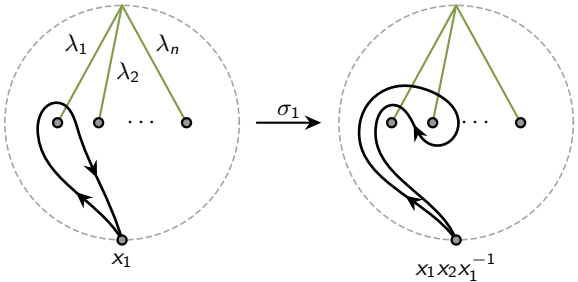
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Lemma

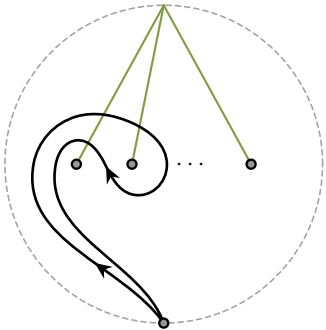
$(\beta \star x_1, \ldots, \beta \star x_n) = \beta^{-1} \cdot (x_1, \ldots, x_n)$. So the Hurwitz words are

$$Q := \mathrm{HurElt}(x_1, \ldots, x_n) = B_n \star x_1 = \{\text{simple loops around one puncture}\}.$$

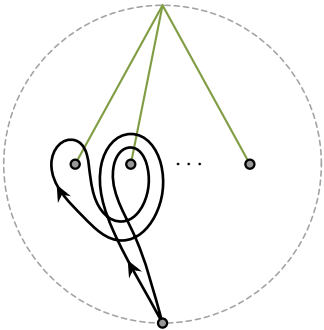
The most important lemma

Lemma

Words in Q correspond to *embedded* (simple) loops in the punctured disk.



$x_1 x_2 x_1^{-1} \in Q$



$x_2 x_1 x_2^{-1} \notin Q$

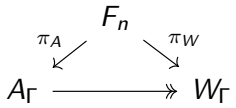
A presentation from Hurwitz words

Let $\pi_A \colon F_n \to A_\Gamma$ and $\pi_W \colon F_n \to W_\Gamma$ be $x_i \mapsto t_i$ and $x_i \mapsto s_i$.

Theorem (O '26; cf. Bessis '04, Resteghini '24)

Let $c = s_1 \cdots s_n$ (any c, after relabelling).

1. $A_\Gamma^\vee(c) \cong \left\langle x_1, \ldots, x_n \;\middle|\; q_1 = q_2 \text{ if } q_1, q_2 \in Q \text{ and } q_1 \overset{W}{=} q_2 \right\rangle$.
2. $t_i \mapsto [s_i]$ defines an epimorphism $\psi \colon A_\Gamma \twoheadrightarrow A_\Gamma^\vee(c)$.
3. ψ is an isomorphism $\iff$ for all $q_1, q_2 \in Q$: $\quad q_1 \overset{W}{=} q_2 \implies q_1 \overset{A}{=} q_2$.



A refinement of the word problem in W_Γ

Goal

For Hurwitz words $q_1, q_2 \in Q$: $\quad q_1 \overset{W}{=} q_2 \implies q_1 \overset{A}{=} q_2$.

Tits' solution to the word problem in W_Γ. Consider two word operations:

$$(1)\ \ s_i s_i \rightsquigarrow \varepsilon, \qquad\qquad (2)\ \ \Pi(s_i, s_j; m_{ij}) \rightsquigarrow \Pi(s_j, s_i; m_{ij}) \quad \text{(Artin move)}.$$

Any word can be reduced using (1) and (2) alone, and reduced words for the same element differ by (2) alone.

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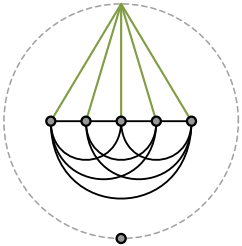
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- Artin moves also hold in A_Γ. The cancellations $s_i s_i \rightsquigarrow \varepsilon$ are the problem.
- A toggle is $\Pi(s_i, s_j; m_{ij} + k) \rightsquigarrow \Pi(s_j, s_i; m_{ij} - k)$. For example, with $m_{ij} = 4$:

$$(s_i s_j s_i s_j) s_i \rightsquigarrow (s_j s_i s_j s_i) s_i \rightsquigarrow s_j s_i s_j.$$

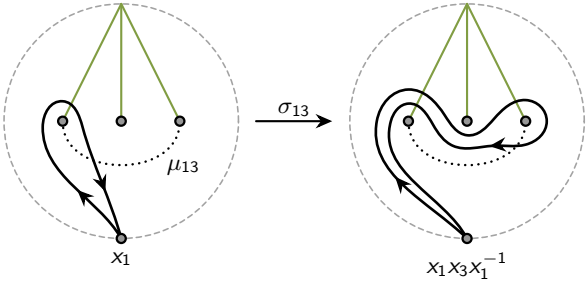
Birman–Ko–Lee generators

- $\sigma_{ij} \in B_n$ is the half twist swapping p_i and p_j along the arc μ_{ij}.



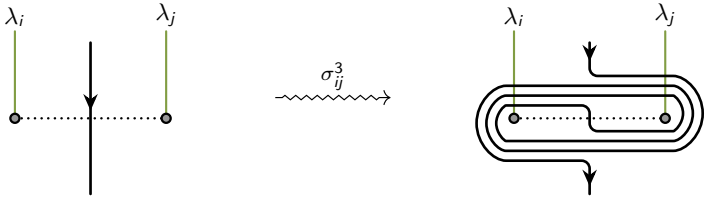
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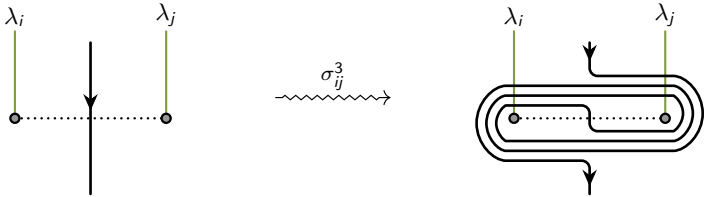
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If $m_{ij} = 3$, this is a relation in A_Γ.

The subgroup $\hat{B}_\Gamma$

Definition (Kluitmann '91)

$\hat{B}_\Gamma := \langle \sigma_{ij}^{m_{ij}} : m_{ij} < \infty \rangle \leq B_n$.

Lemma

$\pi_A(\hat{\beta} \star x) = \pi_A(x)$ for all $\hat{\beta} \in \hat{B}_\Gamma$ and $x \in F_n$.

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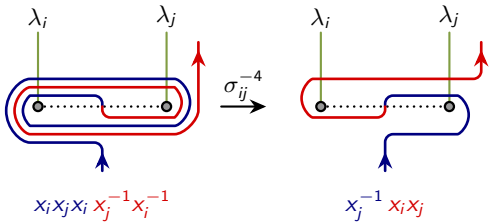
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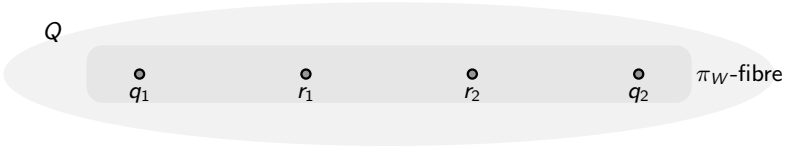
On an (x_i, x_j)-alternating subword, $\sigma_{ij}^{\pm m_{ij}}$ acts in W_Γ as a toggle: length $m_{ij} + k \mapsto m_{ij} - k$.

But it toggles *all* of these subwords at once.



An outline of the proof

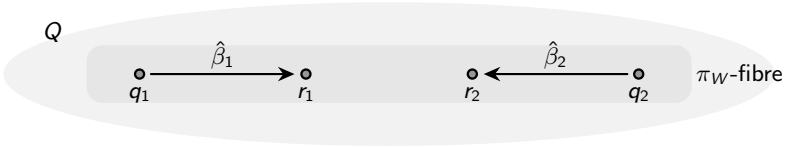
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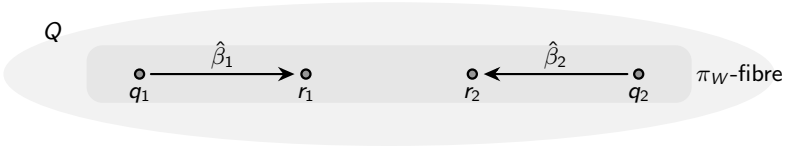
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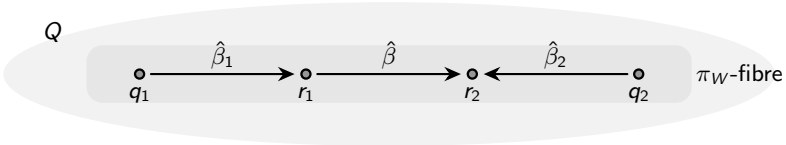
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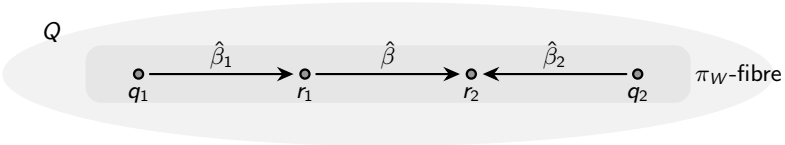


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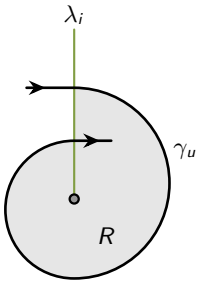
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$$q_2 = \left(\hat{\beta}_2^{-1}\,\hat{\beta}\,\hat{\beta}_1\right) \star q_1 \quad \Longrightarrow \quad q_1 \overset{A}{=} q_2. \qquad \blacksquare$$



How do we use "the most important lemma"

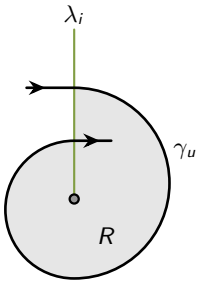


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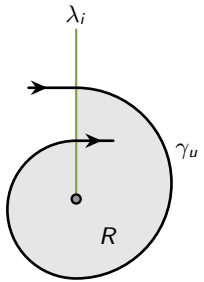
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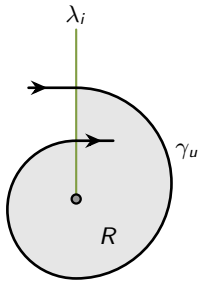
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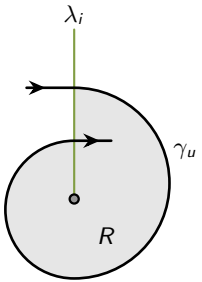
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- So q contains x_i^3. Repeat forever: impossible. ■

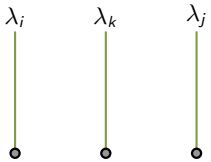
So any cancellation in W_Γ has to be *created* by Artin moves.

Paired exclusion

Lemma (Kluitmann '91)

Let $q \in Q$ and $i \neq j$. Then $x_i^{\pm 1} x_k^{\pm 1} x_i^{\pm 1}$ and $x_j^{\pm 1} x_k^{\pm 1} x_j^{\pm 1}$ are not both subwords of q.

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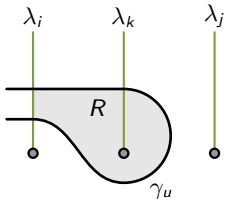
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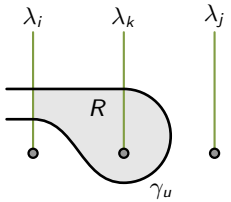
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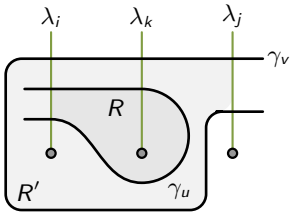
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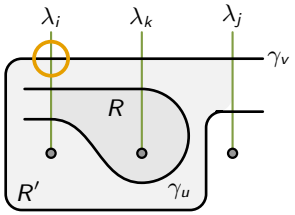
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As a consequence two alternating subwords of length ≥ 4 never share exactly one generator.

Summary

Theorem (O '26)

If all $m_{ij} \geq 5$ (with ≥ 4 very soon), then $A_\Gamma \cong A_\Gamma^\vee(c)$ canonically, for every Coxeter element c.

- The dual Artin isomorphism is the word problem in W_Γ, restricted to Hurwitz words.
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